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INEQUALITIES FOR THE SEIFFERT'S MEANS IN TERMS OF THE IDENTRIC MEAN

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Abstract

Some inequalities for certain bivariate means are obtained. In particular, inequalities for those introduced by Seiffert.

1. Introduction

For $a, b > 0$ with $a \neq b$, the first and second Seiffert's means $P(a, b)$ and $T(a, b)$ were introduced by Seiffert [15, 17] as follows:

$$P(a, b) = \frac{a-b}{4\arctan(\sqrt{a/b}) - \pi}. \tag{1.1}$$

$$T(a, b) = \frac{a-b}{2\arctan\frac{a-b}{a+b}}. \tag{1.2}$$

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Recently, the inequalities for means have been the subject of intensive research [1-14, 16, 18-21]. In particular, many remarkable inequalities for the Seiffert's means can be found in the literature [4, 6-8, 11-13].

Let $H(a, b) = 2ab / (a + b)$, $A(a, b) = (a + b) / 2$, $G(a, b) = \sqrt{ab}$, $I(a, b) = 1 / e(b^b / a^a)^{1/(b-a)}$, and $L(a, b) = (b - a) / (\log b - \log a)$ be the harmonic, arithmetic, geometric, identric, and logarithmic means of two positive real numbers a and b with $a \neq b$. Then,

$$\min\{a, b\} < H(a, b) < G(a, b) < L(a, b) < I(a, b) < A(a, b) < \max\{a, b\}. \tag{1.3}$$

In [15], Seiffert proved

$$L(a, b) < P(a, b) < I(a, b),$$

for all $a, b > 0$ with $a \neq b$.

The following bounds for the first Seiffert's mean $P(a, b)$ in terms of the power mean $M_r(a, b) = ((a^r + b^r)/2)^{1/r} (r \neq 0)$ were presented by Jagers in [8]:

$$M_{1/2} < P(a, b) < M_{2/3}(a, b), \tag{1.4}$$

for all $a, b > 0$ with $a \neq b$.

Hästö [7] found the sharp lower bound for the first Seiffert's mean as follows:

$$M_{\log 2/\log \pi}(a, b) < P(a, b), \tag{1.5}$$

for all $a, b > 0$ with $a \neq b$.

In [16], Seiffert proved

$$P(a, b) > \frac{3A(a, b)G(a, b)}{A(a, b) + 2G(a, b)} \quad \text{and} \quad P(a, b) > \frac{2}{\pi} A(a, b), \tag{1.6}$$

for all $a, b > 0$ with $a \neq b$.

In [4], the authors found the greatest value α and the least value β such that the double inequality $\alpha A(a, b) + (1 - \alpha)H(a, b) < P(a, b) < \beta A(a, b) + (1 - \beta)H(a, b)$ holds for all $a, b > 0$ with $a \neq b$.

In [14], the authors proved

$$I < A < \frac{e}{2} I, \quad A_{2/3} < I < \frac{2\sqrt{2}}{e} A_{2/3}. \tag{1.7}$$

The purpose of the present paper is to obtain the inequalities of type (1.7) for the Seiffert's means in terms of the identric mean.

2. Main Results

In what follows, we will assume, without loss of generality, that $a > b > 0$.

Theorem 2.1. *For the first Seiffert's mean, the double inequality*

$$\frac{e}{\pi} I(a, b) < P(a, b) < I(a, b)$$

holds, where the constants $\frac{e}{\pi}$ *and* 1 *are the best possible.*

Proof. Let $t^2 = a / b > 1$. Consider the function

$$f(t) = \frac{P(t^2, 1)}{I(t^2, 1)} = \frac{e(t^2 - 1)t^{\frac{2t^2}{1-t^2}}}{4 \arctan t - \pi}. \tag{2.1}$$

Its logarithmic derivative is

$$\frac{f'(t)}{f(t)} = \frac{4t \ln t}{(t^2 - 1)^2 (4 \arctan t - \pi)} g(t), \tag{2.2}$$

where

$$g(t) = 4 \arctan t - \pi - \frac{(t^2 - 1)^2}{t(t^2 + 1) \ln t}. \tag{2.3}$$

Simple computations lead to

$$\lim_{t\to 1^+} g(t) = 0, \tag{2.4}$$

$$g'(t) = \frac{g_1(t)}{t^2(t^2+1)^2 \ln^2 t}, \tag{2.5}$$

where

$$g_1(t) = 4t^2(t^2+1)\ln^2 t - (t^6 + 5t^4 - 5t^2 - 1)\ln t + (t^2-1)^2(t^2+1), \tag{2.6}$$

$$\lim_{t\to 1^+} g_1(t) = 0, \tag{2.7}$$

$$g_1'(t) = 8(2t^3 + t)\ln^2 t + 6(-t^5 - 2t^3 + 3t)\ln t + 5t^5 - 9t^3 + 3t + \frac{1}{t}, \tag{2.8}$$

$$\lim_{t\to 1^+} g_1'(t) = 0, \tag{2.9}$$

$$g_1''(t) = 8(6t^2+1)\ln^2 t + (-30t^4 - 4t^2 + 34)\ln t + 19t^4 - 39t^2 + 21 - \frac{1}{t^2}, \tag{2.10}$$

$$\lim_{t\to 1^+} g_1''(t) = 0, \tag{2.11}$$

$$g_1'''(t) = 2tg_2(t), \tag{2.12}$$

where

$$g_2(t) = 48\ln^2 t + (-60t^2 + 44 + \frac{8}{t^2})\ln t + 23t^2 - 41 + \frac{17}{t^2} + \frac{1}{t^4}, \tag{2.13}$$

$$\lim_{t\to 1^+} g_2(t) = 0, \tag{2.14}$$

$$g_2'(t) = \frac{2}{t} g_3(t), \tag{2.15}$$

$$g_3(t) = 48\ln t + (-60t^2 - \frac{8}{t^2})\ln t - 7t^2 + 22 - \frac{13}{t^2} - \frac{2}{t^4}, \tag{2.16}$$

$$\lim_{t\to 1^+} g_3(t) = 0, \tag{2.17}$$

$$g_3'(t) = 2tg_4(t), \tag{2.18}$$

where

$$g_4(t) = \left(-60 + \frac{8}{t^4}\right)\ln t - 37 + \frac{24}{t^2} + \frac{9}{t^4} + \frac{4}{t^6}, \tag{2.19}$$

$$\lim_{t\to 1^+} g_4(t) = 0, \tag{2.20}$$

$$g_4'(t) = \frac{4}{t^5}\left(-8\ln t - 15t^4 - 12t^2 - 7 - \frac{6}{t^2}\right) < 0, \tag{2.21}$$

for $t > 1$, hence $g_4(t)$ is strictly decreasing in $[1, +\infty)$. It follows from (2.20) and (2.18) together with the monotonicity of $g_4(t)$ that $g_3'(t) < 0$, hence $g_3(t)$ is strictly decreasing in $[1, +\infty)$. From (2.17) and (2.15) together with the monotonicity of $g_3(t)$, we know that $g_2'(t) < 0$, hence $g_2(t)$ is strictly decreasing in $[1, +\infty)$.

Repeating the above procedures, we can get $g'(t) < 0$, hence $g(t)$ is strictly decreasing in $[1, +\infty)$.

From (2.4) and (2.2) together with the monotonicity of $g(t)$, we know that $f'(t) < 0$, hence $f(t)$ is strictly decreasing in $[1, +\infty)$.

Hence

$$f(t) < \lim_{t\to 1^+} f(t) = 1,$$

and

$$f(t) > \lim_{t\to +\infty} f(t) = \frac{e}{\pi}.$$

The proof of the inequality $\frac{e}{\pi} I(a, b) < P(a, b) < I(a, b)$ is complete. Since $f(t)$ is continuous for $t > 1$, it follows that the constants $\frac{e}{\pi}$ and 1 are the best possible. $\square$

Theorem 2.2. *For the second Seiffert's mean, the double inequality*

$$I(a, b) < T(a, b) < \frac{2e}{\pi} I(a, b)$$

holds, where the constants 1 *and* $\frac{2e}{\pi}$ *are the best possible.*

Proof. Let $t = a / b > 1$. Consider the function

$$f(t) = \frac{T(t, 1)}{I(t, 1)} = \frac{e(t-1)}{2t^{\frac{t}{t-1}} \arctan \frac{t-1}{t+1}}. \tag{2.22}$$

Its logarithmic derivative is

$$\frac{f'(t)}{f(t)} = \frac{\ln t}{(t-1)^2 \arctan \frac{t-1}{t+1}} g(t), \tag{2.23}$$

where

$$g(t) = \arctan \frac{t-1}{t+1} - \frac{(t-1)^2}{(t^2+1)\ln t}. \tag{2.24}$$

Simple computations lead to

$$\lim_{t \to 1^+} g(t) = 0, \tag{2.25}$$

$$g'(t) = \frac{g_1(t)}{(t^2+1)^2 \ln^2 t}, \tag{2.26}$$

where

$$g_1(t) = (t^2+1)\ln^2 t - 2(t^2-1)\ln t + (t-1)^2 (t + \frac{1}{t}), \tag{2.27}$$

$$\lim_{t \to 1^+} g_1(t) = 0, \tag{2.28}$$

$$g_1'(t) = 2t \ln^2 t + (-2t + \frac{2}{t}) \ln t + 3t^2 - 6t + 2 + \frac{2}{t} - \frac{1}{t^2}, \tag{2.29}$$

$$\lim_{t \to 1^+} g_1'(t) = 0, \tag{2.30}$$

$$g_1''(t) = 2\ln^2 t + (2 - \frac{2}{t^2})\ln t + 6t - 8 + \frac{2}{t^3}, \tag{2.31}$$

$$\lim_{t \to 1^+} g_1''(t) = 0, \tag{2.32}$$

$$g_1'''(t) = \frac{1}{t} g_2(t), \tag{2.33}$$

where

$$g_2(t) = 4\ln t + \frac{4}{t^2}\ln t + 6t - 2 - \frac{2}{t^2} - \frac{6}{t^3}, \tag{2.34}$$

$$\lim_{t \to 1^+} g_2(t) = 0, \tag{2.35}$$

$$g_2'(t) = \frac{1}{t^3} g_3(t), \tag{2.36}$$

$$g_3(t) = 4t^2 - 8\ln t + 8 + 6t^3 + \frac{18}{t}, \tag{2.37}$$

$$\lim_{t \to 1^+} g_3(t) > 0, \tag{2.38}$$

$$g_3'(t) = 8t - \frac{8}{t} + 18t^2 - \frac{18}{t^2}, \tag{2.39}$$

$$\lim_{t \to 1^+} g_3'(t) = 0, \tag{2.40}$$

$$g_3''(t) = 8 + \frac{8}{t^2} + 36t + \frac{36}{t^3}.$$

We can see clearly that $g_3''(t) > 0$ for $t > 1$, hence $g_3'(t)$ is strictly increasing in $[1, +\infty)$. From (2.40), we have $g_3'(t) > 0$ for $t > 1$, hence $g_3(t)$ is strictly increasing in $[1, +\infty)$.

It follows from (2.38) and (2.36) together with the monotonicity of $g_3(t)$ that $g_2'(t) > 0$, hence $g_2(t)$ is strictly increasing in $[1, +\infty)$. From (2.35) and (2.33) together with the monotonicity of $g_2(t)$, we know that $g_1'''(t) > 0$, hence $g_1''(t)$ is strictly increasing in $[1, +\infty)$.

Repeating the above procedures, we can get $g'(t) > 0$, hence $g(t)$ is strictly increasing in $[1, +\infty)$.

From (2.25) and (2.23) together with the monotonicity of $g(t)$, we know that $f'(t) > 0$, hence $f(t)$ is strictly increasing in $[1, +\infty)$.

Hence

$$f(t) > \lim_{t \to 1^+} f(t) = 1,$$

and

$$f(t) < \lim_{t \to +\infty} f(t) = \frac{2e}{\pi}.$$

The proof of the inequality $I(a, b) < P(a, b) < \frac{2e}{\pi} I(a, b)$ is complete. Since $f(t)$ is continuous for $t > 1$, it follows that the constants 1 and $\frac{2e}{\pi}$ are the best possible. □

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